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ALMOST-PERIODIC RESONANCE OSCILLATIONS IN NON-LINEAR TWO-DIMENSIONAL SYSTEMS WITH SLOWLY VARYING PARAMETERS†

V. Sh. BURD

Yaroslavl'

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Almost periodically perturbed two-dimensional systems with rapidly changing phase and slowly varying coefficients are considered. The conditions for the existence and stability of almost-periodic resonance solutions are investigated. Forced oscillations of a mathematical pendulum acted upon by the sum of two small periodic forces with close frequencies are considered as an example.

The resonance modes of non-linear systems which contain fast and slow variables have been investigated in many publications. A method of investigating the steady resonance modes in systems with rapidly changing phases of a general type was proposed in [1]. A formalism of the method of averaging for investigating resonance modes in systems with slowly varying coefficients was developed in [2]. Periodic perturbations of two-dimensional systems with rapidly changing phase and slowly varying coefficients were studied in [3]. The conditions for the accurate and averaged equations to be close to one another in a finite asymptotically long time interval were indicated.

1. Consider the following system of differential equations

$$\dot{x} = \varepsilon f(x, \varphi, \psi, \tau, \varepsilon), \quad \dot{\varphi} = \omega(x, \tau) + \varepsilon g(x, \varphi, \psi, \tau, \varepsilon) \tag{1.1}$$

where

$$\dot{\psi} = \Omega(\tau), \quad \tau = \varepsilon t$$

Here $x(t)$, $\varphi(t)$, $\psi(t)$ are scalar functions, $\tau \in (-\infty, \infty)$ is the slow time, $\varepsilon \in [0, \varepsilon_0)$ is a small parameter, and the dot denotes a derivative with respect to t. The functions $f(x, \varphi, \psi, \tau, \varepsilon)$, $g(x, \varphi, \psi, \tau, \varepsilon)$ are sufficiently smooth with respect to the variables x and φ in a certain region D of the x, φ plane, sufficiently smooth with respect to the parameter ε and almost-periodic functions with respect to each of the variables ψ and τ uniformly with respect to the remaining variables The function $\omega(x, \tau)$ is sufficiently smooth with respect to the variable x in a certain interval and an almost-periodic function with respect to τ uniformly with respect to x. The almost-periodic function $\Omega(\tau)$ is separated from zero

$$\inf_{-\infty<\tau<\infty} |\Omega(\tau)| \neq 0 \tag{1.2}$$

We will call the almost-periodic function $f(t)$ regular if

$$\int_0^t f(s)ds = t\langle f\rangle + r(t)$$

where $\langle f\rangle$ is the mean value of the almost-periodic function $f(t)$ and $r(t)$ is an almost-periodic function. Henceforth we shall assume that $\Omega(\tau)$ is a regular almost periodic function.

System (1.1) is a system with two slow variables x and τ and two fast variables φ and ψ. We will investigate the case of resonance: an almost-periodic function $x_0(\tau)$ exists such that

$$\omega(x_0(\tau), \tau) \equiv 0 \tag{1.3}$$

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and the following conditions of non-degeneracy of the resonance holds

$$\inf_{-\infty<\tau<\infty} |\,\omega_x(x_0(\tau),\tau)| \neq 0 \tag{1.4}$$

($f_x, f_{xx}, \ldots$ are the partial derivatives of the function f with respect to x).

2. We will investigate the behaviour of the solutions of system (1.1) in $\mu = \sqrt{\varepsilon}$—the neighbourhood of the resonance point $x_0(\tau)$.

We put

$$x = x_0(\tau) + \mu z$$

and expand the right-hand side of the converted system in powers of μ. We obtain

$$\begin{aligned} z^{\cdot} &= \mu[f(x_0,\varphi,\psi,\tau,0) - x_{0\tau}] + \mu^2 f_x(x_0,\varphi,\psi,\tau,0)z + O(\mu^3) \\ \varphi^{\cdot} &= \mu\omega_x(x_0,\tau)z + \frac{1}{2}\mu^2\omega_{xx}(x_0,\tau)z^2 + \mu^2 g(x_0,\varphi,\psi,\tau,0) + O(\mu^3) \end{aligned} \tag{2.1}$$

System (2.1) contains only a single fast variable ψ. In system (2.1) we make a substitution, which is standard for the averaging method, which enables us to get rid of the fast variable on the right-hand side of (2.1) apart from terms of the order of μ^2. We will seek this substitution in the form

$$\begin{aligned} z &= \xi + \mu u_1(\eta, \psi, \tau) + \mu^2 u_2(\eta, \psi, \tau)\xi \\ \varphi &= \eta + \mu^2 \upsilon_2(\eta, \psi, \tau) \end{aligned} \tag{2.2}$$

where $u_i(\eta, \psi, \tau)$ $(i = 1, 2)$, $\upsilon_2(\eta, \psi, \tau)$ are defined as the almost-periodic functions in ψ with zero mean from the equations

$$\begin{aligned} \Omega(\tau)\frac{\partial u_1}{\partial \psi} &= f(x_0,\eta,\psi,\tau,0) - f_0(\eta,\tau) \\ \Omega(\tau)\frac{\partial u_2}{\partial \psi} &= f_x(x_0,\eta,\psi,\tau,0) - \frac{\partial u_1}{\partial \eta}\omega_x(x_0,\tau) - f_1(\eta,\tau) \\ \Omega(\tau)\frac{\partial u_2}{\partial \psi} &= g(x_0,\eta,\psi,\tau,0) - \omega_x(x_0,\tau)u_1 - g_0(\eta,\tau) \end{aligned}$$

where the functions $f_0(\eta, \tau), f_1(\eta, \tau), g_0(\eta, \tau)$ are the mean values with respect to ψ of the functions $f(x_0, \eta, \psi, \tau, 0), f_x(x_0, \eta, \psi, \tau, 0), g(x_0, \eta, \psi, \tau, 0)$, respectively, and these three functions are regular almost periodic functions of ψ. The substitution (2.2) gives a system which, in a time τ, is singularly perturbed

$$\begin{aligned} \mu\frac{d\xi}{d\tau} &= f_0(\eta,\tau) - \frac{dx_0}{d\tau} + \mu f_1(\eta,\tau)\xi + O(\mu^2) \\ \mu\frac{d\eta}{d\tau} &= \omega_x(x_0,\tau)\xi + \mu g_0(\eta,\tau) + \frac{1}{2}\mu\omega_{xx}(x_0,\tau)\xi^2 + O(\mu^2) \end{aligned} \tag{2.3}$$

Suppose an almost periodic function $\varphi_0(\tau)$ exists such that

$$f_0(\varphi_0(\tau), \tau) \equiv dx_0/d\tau \tag{2.4}$$

Then the degenerate system (2.3) ($\mu = 0$) has the solution

$$\xi = 0, \quad \eta = \varphi_0(\tau) \tag{2.5}$$

Linearizing the right-hand side of the degenerate system on the solution (2.5), we obtain the matrix

$$A_0(\tau)=\begin{Vmatrix} 0 & f_{0\eta}(\varphi_0(\tau),\tau) \\ \omega_x(x_0,\tau) & 0 \end{Vmatrix}$$

We will assume that the almost-periodic function

$$m(\tau)=\omega_x(x_0,\tau)f_{0\eta}(\varphi_0(\tau),\tau)$$

is sign-constant. Suppose initially (σ_0 = const) that

$$m(\tau)>\sigma_0>0, \quad \tau\in(-\infty,\infty) \tag{2.6}$$

Then the matrix $A_0(\tau)$ for all τ has real eigenvalues of different signs. In this case, as we know [4], in the system

$$\mu\frac{du}{d\tau}=A_0(\tau)u \tag{2.7}$$

for sufficiently small μ, the space of solutions $U(\mu)$ can be represented in the form

$$U(\mu)=U_+(\mu)+U_-(\mu)$$

For the solutions $u_+(\tau,\mu)\in U_+(\mu)$ the inequality $|u_+(\tau,\mu)|\leqslant M_+\exp[-\gamma_+\mu^{-1}\times(\tau-s)]\,|u_+(s,\mu)|$ $(-\infty<s<\tau<\infty)$ is satisfied, while for the solutions $u_-(\tau,\mu)\in U_-(\mu)$ the inequality $|u_-(\tau,\mu)|\leqslant M_-\exp[-\gamma_-\mu^{-1}(\tau-s)]\,|u_-(s,\mu)|$ $(-\infty<\tau<s<\infty)$ is satisfied. Here $M_+, M_-, \gamma_+, \gamma_-$ are positive constants and $|\cdot|$ is a certain norm in R^2.

It follows from the limits of the solutions of system (2.7) that the solution of this system is unstable for sufficiently small μ if the space $X_-(\mu)$ of the initial conditions of the solutions from $U_-(\mu)$ is non-trivial.

We will denote by B the Banach space of the almost-periodic functions with values in R^2 with the usual norm. It follows from the above discussion that the differential operator

$$L(\mu)u=du/d\tau-\mu^{-1}A_0(\tau)u$$

is continuously invertible in B for sufficiently small μ and, consequently, the inhomogeneous system

$$\mu u^{\cdot}=A_0(\tau)u+f(\tau), \quad f(\tau)\in B \tag{2.8}$$

has a unique almost-periodic solution

$$u(\tau,\mu)=L^{-1}(\mu)f(\tau)=\frac{1}{\mu}\int_{-\infty}^{\infty}K(\tau,s,\mu)f(s)ds$$

where

$$|K(\tau,s,\mu)|\leq M\exp[-\gamma\mu^{-1}|\tau-s|] \quad (-\infty<\tau,s<\infty) \tag{2.9}$$

while M and γ are positive constants.

We substitute $\upsilon=\eta(\tau)-\varphi_0(\tau)$ into system (2.3) and write the system obtained in vector form

$$\mu w^{\cdot}=A_0(\tau)w+F(w,\psi,\tau,\mu) \quad (w=(\xi,\upsilon)) \tag{2.10}$$

The following inequality is obviously satisfied

$$|F(0,\psi,\tau,\mu)|\leqslant\omega_1(\mu) \tag{2.11}$$

where $\omega_1(\mu)\to 0$ as $\mu\to 0$. Further, in view of the smoothness of the vector function $F(w,\psi,\tau,\mu)$ with respect to w the following inequality holds

$$| F(w_1,\psi,\tau,\mu)-F(w_2,\psi,\tau,\mu)| \leqslant \omega_2(\rho,\mu)| w_1-w_2| \tag{2.12}$$

where $| w_1 |, | w_2 | \leqslant \rho$, where $\omega_2(\rho, \mu) \to 0$ as $\rho \to 0$, $\mu \to 0$.

The problem of the almost-periodic solution of system (2.10) is equivalent to the problem of the solvability of the following operator equation in space B

$$w(\tau,\mu)=\Pi(w,\mu)=\frac{1}{\mu}\int_{-\infty}^{\infty} K(\tau,s,\mu)F(w,s,\mu)ds \tag{2.13}$$

It follows from inequalities (2.9), (2.11) and (2.12) that numbers a_0 and μ_1 exist such that when $0 < \mu \leqslant \mu_1$ the operator $\Pi(w, \mu)$ on the sphere $\| w \| \leqslant a_0$ of the space B satisfies the conditions of the principle of contractive mappings, where, when $\mu \to 0$, the numbers $a_0(\mu) \to 0$. Hence, the operator equation (2.13) has a unique solution $\| w(\tau, \mu)$ in the sphere $w \| \leqslant a_0$ which approaches (0, 0) uniformly with respect to τ as $\mu \to 0$. From inequality (2.6) and the theorem on stability with respect to the first approximation we obtain approximately that for sufficiently small μ the almost-periodic solution $w(\tau, \mu)$ of system (2.10) is unstable.

We will formulate the result obtained as it applies to system (1.1).

Theorem 1. Suppose an almost-periodic function $x_0(\tau)$ exists, which satisfies equality (1.3) and inequality (1.4). Suppose the regular almost-periodic function $\Omega(\tau)$ satisfies inequality (1.2). Suppose $f(x_0, \varphi, \psi, \tau, \mu)$ is the regular almost-periodic function ψ and an almost-periodic function $\varphi_0(\tau)$ exists which satisfies Eq. (2.4), and inequality (2.6) is satisfied. Then in the $\sqrt{\varepsilon}$-neighbourhood of the resonance point $x_0(\tau)$ a solution of system (1.1) exists which is almost-periodic with respect to t for sufficiently small ε and is unstable.

3. We will first assume that instead of inequality (2.6) the following inequality is satisfied

$$m(\tau)<\sigma_1<0, \quad \tau\in(-\infty,\infty), \quad (\sigma_1=\text{const}) \tag{3.1}$$

In this case the eigenvalues of the matrix $A_0(\tau)$ are pure imaginary for all τ.

We will write system (2.3) in the time t

$$\begin{aligned} \xi^{\cdot} &=\mu\left[f_0(\eta,\tau)-\frac{dx_0}{d\tau}\right]+\mu^2 f_1(\eta,\tau)\xi+O(\mu^3) \\ \eta^{\cdot} &=\mu\omega_x(x_0,\tau)\xi+\mu^2 g_0(\eta,\tau)+\frac{1}{2}\mu^2\omega_{xx}(x_0,\tau)\xi^2+O(\mu^3) \end{aligned} \tag{3.2}$$

In system (3.2) we make the substitution

$$\begin{aligned} \xi &=\mu u(t)+\mu\xi_0(\tau)+\mu^3 u_3(\psi,\tau) \\ \eta &=\varphi_0(\tau)+\mu v(t)+\mu^2 v_0(\tau) \end{aligned}$$

where the almost-periodic function $\varphi_0(\tau)$ is the solution of Eq. (2.4) while the almost-periodic functions $\xi_0(\tau)$ and $\upsilon_0(\tau)$ are the solutions of the equations

$$\begin{aligned} \frac{d\varphi_0}{d\tau} &=\omega_x(x_0,\tau)\xi_0(\tau)+g_0(\varphi_0(\tau),\tau) \\ \frac{d\xi_0}{d\tau} &=f_{0\eta}(\varphi_0,\tau)v_0(\tau)+f_1(\varphi_0,\tau)\xi_0+\langle f_x(x_0,\varphi_0,\psi,\tau,0)u_1(\varphi_0,\psi,\tau)\rangle+ \\ &+\langle f_\varphi(x_0,\varphi_0,\psi,\tau,0)v_2(\varphi_0,\psi,\tau)\rangle+\langle f_\varepsilon(x_0,\varphi_0,\psi,\tau,0)\rangle \end{aligned}$$

respectively. These equations are solvable by virtue of inequalities (1.4) and (2.6). The function $u_3(\psi, \tau)$ is found from an equation which is similar to the equations defining the functions $u_i(\eta, \psi, \tau)$ ($i = 1, 2$).

(The formalism of substitutions of this kind is described in detail in [3].) As a result we obtain the following system of equations

$$u' = \mu a(\tau)v + \mu^2[b(\tau)u + e(\tau)v^2] + O(\mu^3)$$
$$v' = \mu c(\tau)u + \mu^2 d(\tau)v + O(\mu^3) \tag{3.3}$$

where

$$a(\tau) = f_{0\eta}(\varphi_0,\tau),\quad b(\tau) = f_1(\varphi_0,\tau),\quad c(\tau) = \omega_x(x_0,\tau)$$
$$d(\tau) = g_0(\varphi_0,\tau),\quad e(\tau) = \frac{1}{2}f_{0\eta\eta}(\varphi_0,\tau)$$

In the new notation, condition (3.1) takes the form

$$m(\tau) = a(\tau)c(\tau) < -\sigma_1 < 0,\qquad \tau \in (-\infty,\infty) \tag{3.4}$$

while the eigenvalues of the matrix $\mu A_0(\tau)$ are defined by the formulae

$$\lambda_{1,2}(\tau,\mu) = \pm i\mu[-m(\tau)]^{1/2}$$

We will reduce system (3.3) to "standard form", i.e. to a form in which the matrix of the first approximation is zero, using the substitutions

$$u = A\cos\chi + B\sin\chi + \mu[-m(\tau)]^{1/2}n(\tau)(B\cos\chi - A\sin\chi) \tag{3.5}$$
$$v = \delta(\tau)(B\cos\chi - A\sin\chi)$$

where

$$n(\tau) = \frac{1}{2}\left[b(\tau) + d(\tau) - \frac{\delta'(\tau)}{\delta(\tau)}\right],\quad \delta(\tau) = \left[-\frac{c(\tau)}{a(\tau)}\right]^{1/2}$$
$$\chi(\tau,\mu) = \frac{1}{\mu}\int_0^\tau [-m(s)]^{1/2}ds$$

where we have assumed that $[-m(\tau)]^{1/2}$ is a regular almost-periodic function. Substitution (3.5) converts system (3.3) to the following system

$$A' = \mu^2 n(\tau)A + \mu^2\Phi_1(A,B,\chi,\tau) + O(\mu^3) \tag{3.6}$$
$$B' = \mu^2 n(\tau)B + \mu^2\Phi_2(A,B,\chi,\tau) + O(\mu^3)$$

Here $\Phi_1(A,B,\chi,\tau)$, $\Phi_2(A,B,\chi,\tau)$ are functions, periodic in χ, with period 2π, which contain terms not lower than quadratic in A and B.

Obviously, the differential operator

$$Lz = dz/d\tau - M(\tau)z\qquad (M(\tau) = n(\tau)E)$$

where E is the identity matrix, is continuously invertible in the space B if the mean value of the almost-periodic function $n(\tau)$ is non-zero and, consequently, the mean of the almost-periodic function $\sigma(\tau) = b(\tau) + d(\tau)$ is non-zero, and the zeroth solution of the system $Lz = 0$ is asymptotically stable if $\langle\sigma(\tau)\rangle < 0$ and unstable if $\langle\sigma(\tau)\rangle > 0$. In this case the system

$$dz/d\tau = M(\tau)z + f(\tau),\quad f(\tau) \in B$$

has the unique almost-periodic solution

$$z(\tau) = \int_{-\infty}^{\infty} G(\tau, s) f(s) ds$$

where

$$| G(\tau,s)| \leqslant M \exp[-\gamma|\tau - s|] \quad (-\infty < \tau, s < \infty), \qquad M, \gamma > 0$$

Writing system (3.6) in vector form ($z = (A, B)$, $\Phi = (\Phi_1, \Phi_2)$) in time τ, we obtain that the problem of the almost periodic solution of system (3.6) is equivalent to the problem of the solvability in space B of the operator equation

$$z(\tau) = \int_{-\infty}^{\infty} G(\tau, s)[\Phi(z, s, \chi) + O(\mu)] ds \tag{3.7}$$

Further, as in the case of Theorem 1, it can be shown that for sufficiently small μ, the operator defined by the right-hand side of Eq. (3.7) satisfies the conditions of the principle of contractive mappings in a certain sphere $\|z\| \leqslant a_1$ of the space B, where $a_1 \to 0$ as $\mu \to 0$. Hence, Eq. (3.7) has a unique solution in this sphere and, consequently, system (3.6), for sufficiently small μ, has a unique almost periodic solution close to (0, 0). The problem of the stability of the almost-periodic solution can be set up using theorems on stability in the first approximation.

We will formulate the assertion obtained as it applies to system (1.1).

Theorem 2. Suppose the almost-periodic functions $x_0(\tau)$, $\Omega(\tau)$ and $\varphi_0(\tau)$ satisfy the conditions of Theorem 1. Suppose inequality (3.4) is satisfied and the functions $f(x_0, \varphi_0, \psi, \tau, 0)$, $f_x(x_0, \varphi_0, \psi, \tau, 0)$, $g(x_0, \varphi_0, \psi, \tau, 0)$, $f_x(x_0, \varphi_0, \psi, \tau, 0)u_1(\varphi_0, \psi, \tau)$, $f_\varphi(x_0, \varphi_0, \psi, \tau, 0)\upsilon_2(\varphi_0, \psi, \tau)$, $f_\varepsilon(x_0, \varphi_0, \psi, \tau, 0)$, $u_{1\eta}(\varphi_0, \psi, \tau)$, $u_{1\tau}(\varphi_0, \psi, \tau)$ are regular almost-periodic functions of ψ, and $a[-m(\tau)]^{1/2}$ is a regular almost periodic function of τ. Suppose, finally, that the mean value of the almost-periodic function $a(\tau) + d(\tau)$ is non-zero.

Then, system (1.1) in the ε-neighbourhood of the resonance point $x_0(\tau)$ for sufficiently small ε has a unique solution, almost periodic in t, which is asymptotically stable if $\langle\sigma(\tau)\rangle < 0$ and unstable if $\langle\sigma(\tau)\rangle > 0$.

4. We will consider some examples. The equation

$$x^{\cdot\cdot} + \Omega^2 \sin x = \varepsilon[\gamma x^{\cdot} + a_1 \sin \omega t + a_2 \sin(\omega t + \varepsilon \Delta t)] \tag{4.1}$$

describes forced oscillations and rotations of a mathematical pendulum acted upon by the sum of two small periodic forces with close frequencies. Here ε is a small parameter and Ω^2, γ, a_1, a_2, ω, Δ are real positive constants. The function

$$f(t, \tau) = a_1 \sin \omega t + a_2 \sin(\omega t + \varepsilon \Delta t)$$

which is periodic in t with period $2\pi/\omega$ and periodic in $\tau = \varepsilon t$ with period $2\pi/\Delta$, can be written in the form

$$f(t, \tau) = E(\tau) \sin(\omega t + \delta(\tau)) \tag{4.2}$$

$$E(\tau) = (a_1^2 + 2a_1 a_2 \cos \Delta\tau + a_2^2)^{1/2}, \ \mathrm{tg}\,\delta(\tau) = \frac{a_2 \sin \Delta\tau}{a_1 + a_2 \cos \Delta\tau}$$

The function $E(\tau)$ is strictly positive is $a_1 \neq a_2$, which will also be assumed.

Suppose the perturbed pendulum undergoes oscillatory motion. We will introduce the action-angle variables (I, θ) into the unperturbed system and we will change from Eq. (4.1) to a system given by the formulae

$$x = 2 \arcsin k\, \mathrm{sn}[2\pi^{-1} K(k)\theta] = X(I, \theta) \tag{4.3}$$

$$x^{\cdot} = 2k\Omega\, \mathrm{cn}[2\pi^{-1} K(k)\theta] = Y(I, \theta)$$

which contains elliptic Jacobi functions and the complete elliptic integral of the first kind.

The function $k = k(I)$ is defined by the equation

$$I = 8\pi^{-1}\Omega[E(k) - (1 - k^2)K(k)],$$

where $E(k)$ is the complete elliptic integral of the second kind. Note that for fixed k and $\theta = \pi\Omega t/(2K(k))$ the first of formulae (4.3) is a solution of the equation of the unperturbed pendulum in the oscillatory case.

As a result of substitution (4.3) we obtain the following system

$$I^{\cdot} = \varepsilon[f(t,\tau) - \gamma Y(I,\theta)]X_\theta(I,\theta) \tag{4.4}$$
$$\theta^{\cdot} = \frac{\pi\Omega}{2K(k)} - \varepsilon[f(t,\tau) - \gamma Y(I,\theta)]\, X_I(I,\theta)$$

We will say that resonance occurs in system (4.4) if

$$\frac{\pi\Omega}{2K(k(I))} = \frac{r}{s}\omega \tag{4.5}$$

where r and s are relatively prime integers. We will denote the solution of Eq. (4.5), if it exists, by I_{rs}. Putting $\theta = \varphi + (r/s)\omega t$, system (4.4) takes the form

$$I^{\cdot} = \varepsilon\left[f(t,\tau) - \gamma Y\left(I, \varphi + \frac{r}{s}\omega t\right)\right]X_\theta\left(I, \varphi + \frac{r}{s}\omega t\right) \tag{4.6}$$
$$\varphi^{\cdot} = \frac{\pi\Omega}{2K(k)} - \frac{r}{s}\omega - \left[f(t,\tau) - \gamma Y\left(I, \varphi + \frac{r}{s}\omega t\right)\right]X_I\left(I, \varphi + \frac{r}{s}\omega t\right)$$

Consequently, I_{rs} is a point of resonance, in the sense indicated at the beginning of this paper, but I_{rs} is independent of τ.

We will use Theorems 1 and 2 to investigate the resonance modes. The calculation of the mean values with respect to t of the terms on the right-hand side of system (4.6) is based on expansion of the elliptic functions in Fourier series. The mean value with respect to t of the first term on the right-hand side of the first equation of system (4.6) is only non-zero when $r = s$ and $s = 2n+1$ ($n = 0, 1, \ldots$). When $r = 1$ and $s = 2n+1$ this mean value is

$$f_0(\varphi,\tau) = \frac{1}{2}E(\tau)a_n(q)\sin[\delta(\tau) - (2n+1)\varphi]$$
$$a_n(q) = \frac{q^{n+1/2}}{1+q^{2n+1}}, \quad q = \exp\left(-\frac{\pi K(k')}{K(k)}\right), \quad k'^2 = 1 - k^2$$

The mean value of the second term on the right-hand side of the first equation of system (4.6) is equal to γI. Hence, the function $\varphi_0(\tau)$ is found from the equation

$$\sin[\delta(\tau) - (2n+1)\varphi] = \frac{2\gamma I_{rs}}{E(\tau)a_n(q)} = A(\tau) \tag{4.7}$$

Since $a_n(q) \to 0$ as $n \to \infty$, Eq. (4.7) can only have a solution for a finite number of values of n. If Eq. (4.7) is solvable, we have

$$\varphi_{0l}(\tau) = \frac{\delta(\tau)}{2n+1} - \frac{(-1)^l \arcsin A(\tau)}{2n+1} - \frac{l\pi}{2n+1}, \quad l = 0, 1, \ldots, 4n+1$$

Calculating the derivative function $f_0(\varphi, \tau)$ at the point $\varphi_0(\tau)$ we obtain

$$f_{0\varphi}(\varphi_0, \tau) = a(\tau) = -(-1)^l \frac{1}{2}E(\tau)(2n+1)a_n(q)[1 - A^2(\tau)]^{1/2}$$

and, consequently, the function $a(\tau)$ is positive when l is odd and negative when l is even. Further, a simple calculation shows that $c(\tau) < 0$ for all τ and $b(\tau) + d(\tau) = -\gamma$.

The following result is obtained from Theorems 1 and 2. If the resonance point $I_{1,\,2n+1}$ corresponds to the solution of Eq. (4.5), then for sufficiently small ε Eq. (4.1) has $2n+1$ unstable resonance solutions, almost-periodic in t, in the ε-neighbourhood of the resonance point, and $2n+1$ asymptotically stable resonance solutions, almost-periodic in t, in the ε-neighbourhood of the resonance point. When $\varepsilon = 0$ those solutions become periodic solutions of the unperturbed equation, defined by the formulae

$$I = I_0, \quad \theta = \frac{\omega}{2n+1} t + \frac{l\pi}{2n+1} \quad (l = 0, \ldots, 4n+1)$$

Note that the results do not change if we assume that $\gamma = \gamma(\tau)$ is an almost periodic function of τ with positive mean value.

Similar results are also obtained in the case of the rotational motions of an unperturbed pendulum, where the resonance points are found from the equation

$$\frac{\pi\Omega}{kK(k)} = \frac{1}{n}\omega$$

In the same way one can investigate the more general pendulum equation

$$x^{\cdot\cdot} + \Omega^2(\tau)\sin x = \varepsilon[\gamma(\tau)x + E(\tau)\sin(\nu + \delta(\tau))] \tag{4.8}$$

Here $\Omega(\tau)$ is the regular almost periodic function which satisfies condition (1.2), $E(\tau)$ and $\delta(\tau)$ are defined by (4.2), $d\nu/dt = \omega(\tau)$, $\omega(\tau)$ is the regular almost-periodic function, which is separate from zero, and $\gamma(\tau)$ is the almost periodic function with positive mean value.

We will consider solutions of the equation

$$x^{\cdot\cdot} + \Omega^2(\tau)\sin x = 0$$

inside a certain subregion of the oscillatory motions for all τ, where the boundary of this subregion is independent of τ. We will change from Eq. (4.8) to a system using substitution (4.3). The resonance points $I_{rs}(\tau)$ are found from the equation

$$\frac{\pi\Omega(\tau)}{2K(k(I_{rs}(\tau))} = \frac{r}{s}\omega(\tau)$$

where r and s are mutually prime integers. Making the substitution $\theta = \varphi + (r/s)\nu$ in the corresponding system and calculating the mean values of the right-hand sides with respect to ν, we obtain that the function $f_0(\varphi, \tau)$ can only be non-zero when $r = 1, s = 2n + 1$. The equation for determining $\varphi_0(\tau)$ takes the form

$$\sin(\delta(\tau) - (2n+1)\varphi) = \frac{2(dI_{rs}(\tau)/d\tau) + 2\gamma(\tau)I_{rs}(\tau)}{E(\tau)a_n(q)}$$

A calculation of the coefficients $a(\tau)$, $b(\tau)$, $c(\tau)$ and $d(\tau)$ gives the same results as in the previous case. Hence, we can make assertions for Eq. (4.8) similar to those obtained for Eq. (4.1).

The above scheme can also be used to investigate resonance modes of a pendulum, the point of suspension of which oscillates along the vertical or horizontal axis as given by

$$\xi = \varepsilon E(\tau)\sin(v + \delta(\tau)), \quad dv/dt = \omega(\tau)$$

where $E(\tau)$, $\delta(\tau)$, $\omega(\tau)$ are periodic or almost-periodic functions.

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REFERENCES

1. VOLOSOV V. M. and MORGUNOV B. I., *The Averaging Method in the Theory of Non-linear Oscillatory Systems*. Izd. Mosk. Gos. Univ., Moscow, 1971.
2. MITROPOL'SKII Yu. A., *Problems of the Asymptotic Theory of Unsteady Oscillations*. Nauka, Moscow, 1964.
3. MORRISON J. A., Resonance behavior of a perturbed system depending on a slow-time parameter. *J. Math. Analysis Appl.* **21**, 1, 79–98, 1968.
4. CHANG K. W. Almost periodic solutions of singularly perturbed systems of differential equations. *J. Different. Equat.* **4**, 2, 300–307, 1968.

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